Non-supersymmetric dualities in 2+1D from mirror symmetry

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duality has a long history in both high energy and condensed matter physics

basic question: given $H_{UV}$, what’s $H_{IR}$?
two famous successes of perturbative inferences:

1. **QCD**: quarks weakly interacting with a SU(3) gauge field in the UV gives rise to a confined phase with broken chiral symmetry
two famous successes of perturbative inferences:

2. **BCS superconductivity**: electrons with a weak attractive interaction (induced by phonon exchange) give rise to a superconducting phase of “broken” electromagnetic symmetry
in both cases, the effective description of the IR physics is simple: Goldstone bosons connecting the IR physics to the UV description was a great success of both theoretical and, especially, experimental work
Duality is the ability to describe the same physics in two (or more) distinct ways.

This approach is generally non-perturbative.

There are two different notions of duality:

1. Exact equivalence of theories

Two distinct Lagrangians produce the same physics at all length scales.

\[ \text{theory}_1 \rightarrow \text{some complicated transformation} \rightarrow \text{theory}_2 \]
remarkably, (certain) fermion interactions are exactly marginal and can be described by the free compact boson with varying radius

\[
\bar{\Psi} i \gamma^\mu \partial_\mu \Psi + \lambda (\bar{\Psi} \gamma^\mu \Psi)^2 \leftrightarrow R^2(\lambda)(\partial^\mu \phi)^2
\]
2. IR equivalence of theories — this is a manifestation of universality

\[ UV_1 \neq UV_2 \]

\[ \text{RG flow} \]

\[ \text{IR} \]
2. IR equivalence of theories

2+1D example: Peskin-Dasgupta-Halperin, a.k.a., charge-vortex duality

\[ \mathcal{L}_{XY} = |(\partial_\mu - iA_\mu)\phi|^2 - m^2|\phi|^2 - |\phi|^4 \]

global $U(1)_A$ symmetry ($A_\mu$ is an external gauge field):

\[ J_\mu = \phi^*i\partial_\mu\phi - \phi i\partial_\mu\phi^* \]

two phases: insulator/superfluid

\[
m^2 > 0 : \text{unbroken } U(1)_A \iff \langle \phi \rangle = 0
\]
\[
m^2 < 0 : \text{broken } U(1)_A \iff \langle \phi \rangle \neq 0
\]
2. IR equivalence of theories

an apparently different theory (the dual vortex theory):

$$\mathcal{L}_{\text{Ab Higgs}} = |(\partial_{\mu} + ia_{\mu})\varphi|^2 - M^2|\varphi|^2 - |\varphi|^4 - \frac{1}{2\pi} \epsilon^{\mu\nu\rho} a_{\mu} \partial_{\nu} A_{\rho} - \frac{1}{4g^2} f_{\mu\nu}$$

“little” $a_{\mu}$ is a dynamical 2+1D gauge field

global $U(1)_A$ symmetry ($A_{\mu}$ is an external gauge field):

$$J_{\mu} = \frac{1}{2\pi} \epsilon^{\mu\nu\rho} \partial_{\nu} a_{\rho}$$

two phases: insulator/superconductor

$M^2 < 0 : \text{unbroken } U(1)_A \iff \langle \varphi \rangle \neq 0$

$M^2 > 0 : \text{broken } U(1)_A \iff \langle \varphi \rangle = 0$
it’s believed these two theories are IR dual: \(|m|, |M| \ll E \ll g^2 \to \infty\)

\[
| (\partial_\mu - i A_\mu) \phi|^2 - m^2 |\phi|^2 - |\phi|^4 \\
\uparrow \\
| (\partial_\mu + i a_\mu) \varphi|^2 - M^2 |\varphi|^2 - |\varphi|^4 - \frac{1}{2\pi} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu A_\rho - \frac{1}{4g^2} f_{\mu\nu}^2
\]

dual realizations of the same phases and intervening critical point:

unbroken \(U(1)_A: m^2 > 0 \leftrightarrow M^2 < 0\)

broken \(U(1)_A: m^2 < 0 \leftrightarrow M^2 > 0\)

in the broken phase, Goldstone is dual to the dynamical gauge field

duality interchanges charges and vortices: vortices in the XY-model map to \(\varphi\), while vortices in the Abelian Higgs model are identified with \(\hat{\varphi}\)
the arguments in this talk use a supersymmetric (SUSY) generalization of charge-vortex duality called **mirror symmetry**

I’ll argue that **mirror symmetry** implies (schematically):

a free fermion is dual to a scalar coupled to a $U(1)_1$ photon

a Wilson-Fisher boson is dual to fermion coupled to a $U(1)_{1/2}$ photon

a Wilson-Fisher boson is dual to scalar coupled to a $U(1)_0$ photon

**In this talk, mirror symmetry will be taken to be an axiom.**

**Why use a SUSY duality?**

SUSY provides some control over the dynamics and enables many non-trivial checks of the mirror symmetry duality.
Where do these dualities arise?

(1) the theoretical description of:
• quantum Hall plateau transitions
• (certain) superconductor-insulator transitions
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\[ \nu \equiv \frac{\text{electron density}}{\text{magnetic field}} \]
Where do these dualities arise?

(1) the theoretical description of:
- **quantum Hall plateau transitions**
- (certain) superconductor-insulator transitions

\[ \nu \equiv \frac{\text{electron density}}{\text{magnetic field}} \]

Shahar, Tsui, Shayegan, Shimshoni, & Sondhi (1997)
Shahar, Tsui, Shayegan, Bhatt, & Cunningham (1995)

![Graph (a)](image1)

Shahar, Tsui, Shayegan, Shimshoni, & Sondhi (1997)

![Graph (b)](image2)
Where do these dualities arise?

(1) the theoretical description of:
- quantum Hall plateau transitions
- (certain) superconductor-insulator transitions

\[ \nu \equiv \frac{\text{electron density}}{\text{magnetic field}} \]

Jiang, Stormer, Tsui, Pfeiffer, & West (1989)
Where do these dualities arise?

(1) the theoretical description of:
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Where do these dualities arise?

(1) the theoretical description of:
• quantum Hall plateau transitions
• (certain) superconductor-insulator transitions

**composite fermions** (and **bosons**) provide both an intuitive and qualitatively successful picture for the physics of the 2DEG in B-field

the physics of an electron in a B-field is traded for the physics of **composite fermions** in reduced magnetic flux

fractional quantum Hall states = integer Hall states of **composite fermions**; metallic behavior is also motivated by this approach

from D. Arovas’ Ph.D thesis
Where do these dualities arise?

(2) within a larger “duality web”:

\[
\begin{align*}
N_f \text{ fermions coupled to } SU(k)_{-N+\frac{N_f}{2}} & \leftrightarrow N_f \text{ scalars coupled to } U(N)_{k,k}; \\
N_f \text{ scalars coupled to } SU(N)_{k} & \leftrightarrow N_f \text{ fermions coupled to } U(k)_{-N+\frac{N_f}{2},-N+\frac{N_f}{2}}; \\
N_f \text{ fermions coupled to } U(k)_{-N+\frac{N_f}{2},-N\mp k+\frac{N_f}{2}} & \leftrightarrow N_f \text{ scalars coupled to } U(N)_{k,k\pm N}.
\end{align*}
\]

Giombi, Minwalla, Prakash, Trivedi, & Wadia; Aharony, Gur-ari, & Yacoby; Aharony; Hsin & Seiberg

I’ll use mirror symmetry to argue for the first two dualities at \( N_f = k = N = 1 \).
structure of the talk:

theory A

phase diagram

theory B

mirror symmetry duality

bosonization dualities
dual RG flow

theory A phase diagram

theory B phase diagram
What’s mirror symmetry?

a conjectured set of IR dualities between distinct SUSY theories in 2+1D

Intriligator & Seiberg

in this talk, I will only study the simplest incarnation:

free $\mathcal{N} = 4$ hypermultiplet $\leftrightarrow \mathcal{N} = 4 U(1)$ gauged hypermultiplet

theory A $\leftrightarrow$ theory B

$\mathcal{N} = 4$ in 3D means the theory preserves 8 real Grassmann super-charges — the same amount of SUSY as a $\mathcal{N} = 2$ theory in 4D
free $\mathcal{N} = 4$ hypermultiplet $\leftrightarrow \mathcal{N} = 4$ $U(1)$ gauged hypermultiplet

theory A $\leftrightarrow$ theory B

(field theoretic) **evidence**: 

- matching moduli space, i.e., matching “phase structure”
  (in general)

  $$\text{Higgs}(A) = \text{Coulomb}(B)$$
  $$\text{Coulomb}(A) = \text{Higgs}(B)$$

  **Intriligator & Seiberg**

- along the Coulomb branch of theory B, theory A can be derived

  **Seiberg & Witten; Sachdev & Yin**

- the monopole operators of the $U(1)$ gauged theory saturate the unitarity (lower) bound

  **Kapustin**

- matching 3-sphere partition functions;
  it’s believed this is equivalent to matching constant sub-leading terms in the disk entanglement entropy

  **Kapustin & Willett**

  **Myers & Sinha, Casini, Huerta, & Myers**
free $\mathcal{N} = 4$ hypermultiplet $\leftrightarrow \mathcal{N} = 4 \ U(1)$ gauged hypermultiplet

theory A $\leftrightarrow$ theory B

I first want to explain the statement of mirror symmetry:

- SUSY building blocks
- dual Lagrangians
- deformations and how they map across duality
- general mirror symmetry statement
free $\mathcal{N} = 4$ hypermultiplet $\leftrightarrow \mathcal{N} = 4 \ U(1)$ gauged hypermultiplet

**SUSY building blocks**

SUSY theories are created out of sets or multiplets of bosonic and partner fermionic fields; interactions are dictated by SUSY

**theory A**

$\mathcal{N} = 4$ hypermultiplet : pair of $\mathcal{N} = 2$ chiral multiplets

$$(v_\pm, \Psi_\pm)$$

**theory B**

$U(1)$ gauged $\mathcal{N} = 4$ hypermultiplet :

$\mathcal{N} = 4$ hypermultiplet : pair of charged $\mathcal{N} = 2$ chiral multiplets

$$+ \quad (u_\pm, \psi_\pm)$$

$\mathcal{N} = 4$ vectormultiplet : $\mathcal{N} = 2$ vectormultiplet + neutral $\mathcal{N} = 2$ chiral multiplet

$$(a_\mu, \sigma, \lambda_\alpha, D) + (\phi, \psi_\phi)$$
Mirror symmetry: theory A and theory B are IR dual, \( E \ll g^2 \to \infty \)

theory A

\[ \mathcal{L}^{(A)} = \sum_{\pm} \left( |\partial_\mu v_\pm|^2 \right) \]

theory B

\[ \mathcal{L}^{(B)} = \mathcal{L}_V + \mathcal{L}_H \]

\[ \mathcal{L}_V = \frac{1}{g^2} \left( - \frac{1}{4} f_{\mu\nu}^2 \right) \]

\[ \mathcal{L}_H = \sum_{\pm} \left( |D_\mu^\pm u_\pm|^2 \right) \]
Mirror symmetry: theory A and theory B are IR dual, $E \ll g^2 \to \infty$

**theory A**

$$\mathcal{L}^{(A)} = \sum_{\pm} \left( |\partial_{\mu} v_{\pm}|^2 + i\bar{\Psi}_\pm \partial \Psi_\pm \right)$$

**theory B**

$$\mathcal{L}^{(B)} = \mathcal{L}_V + \mathcal{L}_H$$

$$\mathcal{L}_V = \frac{1}{g^2} \left( -\frac{1}{4} f_{\mu\nu}^2 + \frac{1}{2} (\partial_{\mu} \sigma)^2 + |\partial_{\mu} \phi|^2 + i\bar{\lambda} \partial \lambda + i\bar{\psi}_\phi \partial \psi_\phi \right)$$

$$\mathcal{L}_H = \sum_{\pm} \left( |D_{\mu}^\pm u_\pm|^2 + i\bar{\psi}_\pm D^\pm \psi_\pm \right)$$
Mirror symmetry: theory A and theory B are IR dual, $E \ll g^2 \rightarrow \infty$

theory A

$$\mathcal{L}^{(A)} = \sum_\pm \left( |\partial_\mu v_\pm|^2 + i\bar{\Psi}_\pm \partial \Psi_\pm \right)$$

theory B

$$\mathcal{L}^{(B)} = \mathcal{L}_V + \mathcal{L}_H + \mathcal{L}_{\text{interactions}}$$

$$\mathcal{L}_V = \frac{1}{g^2} \left( -\frac{1}{4} f_{\mu\nu}^2 + \frac{1}{2} (\partial_\mu \sigma)^2 + |\partial_\mu \phi|^2 + i\bar{\lambda} \partial \lambda + i\bar{\psi}_\phi \partial \psi_\phi + \frac{1}{2} D^2 + |F|^2 \right)$$

$$\mathcal{L}_H = \sum_\pm \left( |D^\pm_\mu u_\pm|^2 + i\bar{\psi}_\pm \partial D^\pm \psi_\pm \right)$$

$$\mathcal{L}_{\text{interactions}} = -\left( \sigma^2 + |\phi|^2 \right) (|u_+|^2 + |u_-|^2) - D(|u_+|^2 - |u_-|^2) + Fu_+u_- - \sigma(\bar{\psi}_+ \psi_+ - \bar{\psi}_- \psi_-) - \phi \psi_+ \psi_- - i\psi_\phi (u_+ \psi_- + u_- \psi_+) - i\lambda (u_+^\dagger \psi_+ - u_-^\dagger \psi_-) + h.c.$$
clearly, theory B is complicated and so we need some guidance

**logic of the approach:**

- theory A and theory B have identical global symmetries
- SUSY enables us to turn on sources in non-dynamical vector multiplets minimally coupled to these global symmetry currents
- these deformations can be mapped consistently across the duality
- first we deform, $\mathcal{N} = 4 \rightarrow \mathcal{N} = 2$, then we break SUSY entirely
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**pertinent** symmetries

<table>
<thead>
<tr>
<th>theory A</th>
<th>$U(1)_R$</th>
<th>$U(1)_A$</th>
<th>$U(1)_J$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nu_+$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$\nu_-$</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$\Psi_+$</td>
<td>0</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$\Psi_-$</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
</tr>
</tbody>
</table>

for applications discussed later, it’s useful to identify: $U_J \equiv U(1)_{\text{EM}}$
we can couple the conserved current of a global symmetry to an external gauge field:

$$\mathcal{L}^{(A)} = \sum_{\pm} \left( |\partial_\mu v_\pm|^2 + i\bar{\Psi}_\pm \partial_\Psi_\pm \right)$$
we can couple the conserved current of a global symmetry to an external gauge field:
we can couple the conserved current of a global symmetry to an external gauge field:

\[
\mathcal{L}^{(A)} = |D_{\hat{A}_J} v_+|^2 + |D_{-\hat{A}_J} v_-|^2 + i\bar{\Psi} + \bar{\mathcal{D}}_{\hat{A}_J} \Psi_+ + i\bar{\Psi} - \mathcal{D}_{-\hat{A}_J} \Psi_-
\]
we can couple the conserved current of a global symmetry to an external gauge field:

$$\mathcal{L}^{(A)} = |D_{\hat{A}_J} v_+|^2 + |D_{-\hat{A}_J} v_-|^2 + i\bar{\Psi}_+ \not{D}_{\hat{A}_J} \Psi_+ + i\bar{\Psi}_- \not{D}_{-\hat{A}_J} \Psi_-$$

the external gauge fields can be included in a $\mathcal{N} = 4$ vectormultiplet

$$\mathcal{L}^{(A)}(Q, \hat{V}_J) = \mathcal{L}^{\mathcal{H}}(Q, \hat{V}_J) = \int d^4\theta \left( V^\dagger_+ e^{2\hat{V}_J} V_+ + V^\dagger_- e^{-2\hat{V}_J} V_- \right) + \int d^2\theta \sqrt{2} i\hat{\Phi}_J V_+ V_- + \text{h.c.}$$

$$Z^{(A)}[\hat{V}_J] = \int DQ \exp \left( i \int d^3x \mathcal{L}^{(A)}(Q, \hat{V}_J) \right)$$

$$Q = V_\pm = (v_\pm, \Psi_\pm)$$

$$\hat{V}_J = (\hat{A}_J, \hat{\sigma}_J, \hat{\lambda}_J, \hat{D}_J) + (\hat{\phi}_J, (\hat{\psi}_\phi)_J)$$
How will we use this?

| |
|---|---|---|
| $U(1)_R$ | $U(1)_A$ | $U(1)_J$ |
| $\nu_+$ | 1 | -1 | 1 |
| $\nu_-$ | 1 | -1 | -1 |
| $\Psi_+$ | 0 | -1 | 1 |
| $\Psi_-$ | 0 | -1 | -1 |

turn on the scalar component of the $\mathcal{N} = 2$ $U(1)_J$ vectormultiplet

$$\hat{V}_J = (\hat{A}_J, \hat{\sigma}_J, \hat{\lambda}_J, \hat{D}_J) = (0, \hat{\sigma}_J, 0, 0)$$

"SUSY minimal coupling" and the charge table says the Lagrangian obtains the terms:

$$\mathcal{L}^{(A)} = -\hat{\sigma}_J^2 |\nu_+|^2 - \hat{\sigma}_J \bar{\Psi}_+ \Psi_+ - (-\hat{\sigma}_J)^2 |\nu_-|^2 - (-\hat{\sigma}_J) \bar{\Psi}_- \Psi_-$$

in this way, we can decouple fields; duality tells us how to map
### (pertinent) symmetries

<table>
<thead>
<tr>
<th>Theory B</th>
<th>$U(1)_R$</th>
<th>$U(1)_A$</th>
<th>$U(1)_J$</th>
<th>$U(1)_a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_+$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$u_-$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>$\psi_+$</td>
<td>-1</td>
<td>1</td>
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</tr>
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<td>$\psi_-$</td>
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</tr>
</tbody>
</table>

$$
f_{\mu\nu} = \epsilon_{\mu\nu\rho} \partial^\rho \gamma$$

$$
J_\mu = \frac{1}{2\pi} \epsilon^{\mu\nu\rho} \partial_\nu a_\rho
$$

$$
\mathcal{L}^{(B)}(U, V, \hat{V}_J) = \mathcal{L}^V(V) + \mathcal{L}^H(U, V) - \mathcal{L}_{BF}^{N=4}(V, \hat{V}_J)
$$

$$
Z^{(B)}[\hat{V}_J] = \int DU DV \exp \left( i \int d^3x \mathcal{L}^{(B)}(U, V, \hat{V}_J) \right)
$$

$U = U_\pm = (u_\pm, \psi_\pm)$

$V = (a_\mu \lambda, \sigma, \psi_\phi, \phi)$
mirror symmetry: theory A and theory B are IR dual

in the limit: \( E \ll g^2 \rightarrow \infty \)

\[
Z^{(A)}[\hat{V}_J, \hat{V}_A] = Z^{(B)}[\hat{V}_J, \hat{V}_A]
\]

\( \hat{V}_{J,A} : \) external \( \mathcal{N} = 2 \) vectormultiplets

(the scale defined by the external source should be less than \( g^2 \) )

How should we deform?

• first, we’ll derive a mirror symmetry duality between \( \mathcal{N} = 2 \) theories

• next, we’ll break SUSY and obtain interesting non-SUSY duals
An important comment regarding a weak point of our argument.

The dynamics of theory A are easily analyzed, being either a free field theory or the Wilson-Fisher fixed point.

Mirror symmetry guarantees that the relevant perturbations added to theory A can be mapped across the duality and result in an identical phase diagram.

Because theory B is strongly coupled, a direct analysis is not possible (this is what theory A is for!).

Nevertheless, we interpret the effects of the relevant perturbations to theory B with respect to its tree-level Lagrangian: denoting the mass scale associated to a relevant perturbation of theory A as $M$, we analyze the effects of the relevant perturbation in the regime

$$g^2 \ll M$$

and assume (and find consistency with the theory A phase diagram!) the results continue to the regime of interest

$$M \ll g^2$$
• a duality between $\mathcal{N} = 2$ theories

theory A

\[
\mathcal{L}^{(A)} = \sum_{\pm} \left( |\partial_\mu v_\pm|^2 + i\bar{\Psi}_\pm \partial \Psi_\pm \right)
\]

turn on background, non-dynamical fields (indicated by “hats”):

\[
|\hat{\sigma}_A - \hat{\sigma}_J| \ll \hat{\sigma}_A \sim \hat{\sigma}_J
\]

\[
0 \leq \hat{D}_J \ll \hat{\sigma}_A
\]

\[
\delta L^{(A)} = - (\hat{\sigma}_A + \hat{\sigma}_J)^2 |v_-|^2 + (\hat{\sigma}_A + \hat{\sigma}_J)\bar{\Psi}_- \Psi_-
\]

the chiralmultiplet $\left(v_-, \Psi_\right)$ decouples at low energies leaving:

\[
\mathcal{L}^{(A)} = |D_{\hat{A}_J} v_+|^2 - \left( (\hat{\sigma}_J - \hat{\sigma}_A)^2 + \hat{D}_J \right) |v_+|^2 + i\bar{\Psi}_+ \hat{D}_{\hat{A}_J} \Psi_+ - (\hat{\sigma}_J - \hat{\sigma}_A)\bar{\Psi}_+ \Psi_+ - \frac{1}{8\pi} \hat{A}_J d\hat{A}_J
\]

this is simply the theory of a single, free chiralmultiplet $\left(v_+, \Psi_+\right)$

stability requires: $$(\hat{\sigma}_J - \hat{\sigma}_A)^2 + \hat{D}_J \geq 0$$
theory B

set \( \hat{\sigma}_J = \hat{\sigma}_A \) and ignore the low-energy SUSY-breaking \( \hat{D}_J \)

\[
\begin{array}{c|cccc}
 & U(1)_R & U(1)_A & U(1)_J & U(1)_a \\
\hline
u_+ & 0 & 1 & 0 & 1 \\
u_- & 0 & 1 & 0 & -1 \\
\psi_+ & -1 & 1 & 0 & 1 \\
\psi_- & -1 & 1 & 0 & -1 \\
\hline
e^{2\pi i \gamma/g^2} & 0 & 0 & 1 & 0 \\
\sigma & 0 & 0 & 0 & 0 \\
\phi & 2 & -2 & 0 & 0 \\
\lambda & 1 & 0 & 0 & 0 \\
\psi_\phi & 1 & -2 & 0 & 0 \\
\end{array}
\]

\[
f_{\mu\nu} = \epsilon_{\mu\nu\rho} \partial^\rho \gamma \\
J_\mu = \frac{1}{2\pi} \epsilon^{\mu\nu\rho} \partial_\nu a_\rho
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theory B

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$$

$$
f_{\mu \nu} = \epsilon_{\mu \nu \rho} \partial^\rho \gamma \\
J_\mu = \frac{1}{2\pi} \epsilon^{\mu \nu \rho} \partial_\nu a_\rho
$$

the neutral chiral multiplet $(\phi, \psi_\phi)$ obtains a mass and is decoupled

$$m_\phi^2 = m_{\psi_\phi}^2 = (-2\hat{\sigma}_A)^2$$

the effect on the “charged” matter fields is more subtle
the “charged” matter fields $U_\pm = (u_\pm, \psi_\pm)$ have masses:

$$m_{U_\pm} = \sigma \pm \hat{\sigma}_A$$

thus, we need to determine $\langle \sigma \rangle$ when $\hat{\sigma}_A = \hat{\sigma}_J > 0$

a SUSY-preserving vacuum (zero vacuum energy) requires that the “FI D-term” vanishes

$$\frac{|\sigma + \hat{\sigma}_A|}{2} - \frac{|\sigma - \hat{\sigma}_A|}{2} = \hat{\sigma}_J$$

one solution:

$$\langle \sigma \rangle = \hat{\sigma}_A \implies U_+ \text{ is massive}$$

in general,

$$\langle \sigma \rangle \geq \hat{\sigma}_A$$

this Coulomb branch of the moduli space corresponds to vevs of $U_+$
interpreting these masses in terms of the UV Lagrangian, the upshot is that we are left with a $\mathcal{N} = 2$ $U(1)$ charged chiral multiplet

$$\mathcal{L}^{(B)} = \mathcal{L}_V + \mathcal{L}_{\text{matter}} + \mathcal{L}_{CS} - \mathcal{L}_{BF}$$

$$\mathcal{L}_V = \frac{1}{g^2} \left( -\frac{1}{4} f_{\mu\nu}^2 + \frac{1}{2} (\partial \sigma)^2 + \bar{\lambda} \, \partial \lambda + \frac{1}{2} D^2 \right)$$

$$\mathcal{L}_{\text{matter}} = |D_{-a} u|^2 + \bar{\psi} i \slashed{D}_{-a} \psi - (\sigma^2 - D)|u|^2$$

$$+ \sigma \bar{\psi} \psi + u^* \bar{\lambda} \psi + u \bar{\psi} \lambda,$$

$$\mathcal{L}_{CS} = \frac{1}{8\pi} (a da + 2 D \sigma + \bar{\lambda} \lambda),$$

$$\mathcal{L}_{BF} = \frac{1}{2\pi} \left( a d \hat{A}_J + \hat{D}_J \sigma + \hat{\sigma}_J D \right)$$

$$\sigma = \text{fluctuations about } \hat{\sigma}_J$$

$$(u_-, \psi_-) \equiv (u, \psi)$$
a new dual pair

N=4 hypermultiplet

N=4 mirror
symmetry duality

N=4 U(1) gauged
hypermultiplet

RG flow

dual RG flow

N=2 chiralmultiplet

N=2 duality

N=2 U(1) gauged
chiralmultiplet
now, we want to break SUSY to derive non-SUSY dualities

N=4 hypermultiplet

N=4 mirror symmetry duality

N=4 U(1) gauged hypermultiplet

RG flow

dual RG flow

N=2 chiralmultiplet

N=2 duality

N=2 U(1) gauged chiralmultiplet
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RG flow

N=2 chiralmultiplet

N=2 duality

N=2 U(1) gauged chiralmultiplet

dual RG flow

RG flow

theory A phase diagram

non-SUSY duality

dual RG flow

classical RG flow

theory B phase diagram
\[ \mathcal{L}^{(A)} = |D_{\hat{A}_J} v|^2 - m_v^2 |v|^2 + i\bar{\Psi} D_{\hat{A}_J} \Psi - m_\Psi \bar{\Psi} \Psi - \frac{1}{8\pi} \hat{A}_J d\hat{A}_J \]

\[ m_v^2 = \hat{\sigma}_J^2 + \hat{D}_J, \]

\[ m_\Psi = \hat{\sigma}_J \]

\[ (V_+ = (v, \Psi)) \]

by tuning \( \hat{\sigma}_J \) with \( \hat{D}_J \geq 0 \), I'll argue for the bosonization duality:

a free fermion is dual to a scalar coupled to a \( U(1)_1 \) photon

\[ \bar{\Psi} i \slashed{\partial} \hat{A} \Psi - \frac{1}{8\pi} \hat{A} d\hat{A} \leftrightarrow |D_{-a} u|^2 - |u|^4 + \frac{1}{4\pi} ada - \frac{1}{2\pi} \hat{A} da \]

\[ \hat{A} = \hat{A}_J \]
theory A

\[ \mathcal{L}^{(A)} = |D_{\hat{A}_J} v|^2 - m_v^2 |v|^2 + i\bar{\Psi} \not D_{\hat{A}_J} \Psi - m_\Psi \bar{\Psi} \Psi - \frac{1}{8\pi} \hat{A}_J d\hat{A}_J \]

\[ m_v^2 = \hat{\sigma}_J^2 + \hat{D}_J, \]

\[ m_\Psi = \hat{\sigma}_J \]

\[ m_v^2 > 0, \]

\[ m_\Psi > 0 \]

I

II

III

- sign(\hat{\sigma}_J)\hat{\sigma}_J^2

\[ m_v^2 < 0 \]
theory A

\[ \mathcal{L}^{(A)} = |D_{\hat{A}_J} \psi|^2 - m_v^2 |\psi|^2 + i \Psi \slashed{D}_{\hat{A}_J} \Psi - m_\Psi \bar{\Psi} \Psi - \frac{1}{8\pi} \hat{A}_J d\hat{A}_J \]

\[ m_v^2 = \hat{\sigma}_J^2 + \hat{D}_J, \]

\[ m_\Psi = \hat{\sigma}_J \]

\[ \dot{D}_J \]

\[ -\text{sign}(\hat{\sigma}_J) \hat{\sigma}_J^2 \]

\[ III \]

\[ m_v^2 < 0 \]
theory A

\[ \mathcal{L}^{(A)} = |D_{A_J} v|^2 - m_v^2 |v|^2 + i \bar{\Psi} \not{D}_{A_J} \Psi - m_\Psi \bar{\Psi} \Psi - \frac{1}{8\pi} \hat{A}_J d\hat{A}_J \]

\[ m_v^2 = \hat{\sigma}_J^2 + \hat{D}_J, \]

\[ m_\Psi = \hat{\sigma}_J \]

\[ \mathcal{L}^{\text{II}} = 0 \]

\[ \mathcal{L}^{\text{I}} = -\frac{1}{4\pi} \hat{A}_J d\hat{A}_J \]

- \text{III}

\[ m_v^2 < 0 \]
theory A

\[ \mathcal{L}^{II} = 0 \]

\[ \mathcal{L}^{I} = -\frac{1}{4\pi} \hat{A}_J d\hat{A}_J \]

\[ m_v^2 < 0 \]
theory A

\[ \hat{D}_J > 0 \]

\[ \mathcal{L}_{\text{eff}}^{(A)} = i \bar{\Psi} \slashed{D} \hat{A}_J \Psi - \hat{\sigma}_J \bar{\Psi} \Psi - \frac{1}{8\pi} \hat{A}_J d \hat{A}_J \]

\[ \mathcal{L}^{\Pi} = 0 \]

\[ \mathcal{L}^{I} = -\frac{1}{4\pi} \hat{A}_J d \hat{A}_J \]

\[ m_v^2 < 0 \]
theory B

dual theories must have **identical** phase structure

the realization can be **distinct**

\[
\mathcal{L}^{\Pi} = 0 \\
\mathcal{L}^{I} = -\frac{1}{4\pi} \hat{A}_J d\hat{A}_J
\]
theory B

First, I want to determine how these two phases are realized in terms of tree-level parameters of theory B.

\[ \text{sign}(\hat{J}) \hat{J}^2 J_L = 1 \]

\[ \lambda A J d A J \]

\[ \mu^2 = ? \]

\[ m_{\psi} = ? \]

\[ m_\lambda = ? \]

\[ m_\sigma^2 = ? \]

Then, I want to argue that when SUSY is broken, only \( u \) becomes massless as we pass from one phase to the another.
theory B

unbroken SUSY and duality implies a unique vacuum when $\hat{D}_J = 0$ and $\hat{\sigma}_J \neq 0$

the usual SUSY analysis is straightforward when $\hat{\sigma}_J < 0$

\[
\begin{align*}
m_u^2 & > 0 \\
m_\psi & > 0 \\
m_\lambda & < 0 \\
m_\sigma^2 & > 0
\end{align*}
\]
theory B
unbroken SUSY and duality imply a unique vacuum when $\hat{D}_J = 0$ and $\hat{\sigma}_J \neq 0$

the vacuum analysis is likewise simple when $\hat{\sigma}_J > 0$
theory B

unbroken SUSY and duality imply a unique vacuum when $\hat{D}_J = 0$ and $\hat{\sigma}_J \neq 0$
theory B

unbroken SUSY and duality imply a unique vacuum when \( \hat{D}_J = 0 \) and \( \hat{\sigma}_J \neq 0 \)

the fermion masses when \( \hat{\sigma}_J > 0 \) ...

the non-zero VEV \( \langle u \rangle \) implies a non-diagonal fermion mass matrix

\[
\delta m \propto \langle u \rangle \\
\implies \\
\mathcal{L}^{(B)}_{\text{mass}} = -m_{\psi} \bar{\psi}\psi - m_\lambda \bar{\lambda}\lambda + \delta m^* \bar{\lambda}\psi + \delta m \bar{\psi}\lambda
\]

with eigenvalues

\[
m_{f_\pm} = \frac{1}{2} \left( m_\psi + m_\lambda \pm \sqrt{(m_\psi - m_\lambda)^2 + 4|\delta m|^2} \right)
\]

SUSY implies

\[
m_{f_+} > 0, m_{f_-} < 0
\]

we’ll argue: as \( \delta m \to 0 \), \( f_+ \to \psi \), \( f_- \to \lambda \)
the SUSY analysis implies that within the two phases along \( \hat{D}_J = 0 \)

\[
\mathcal{L}_{\text{eff}}^I = \frac{1}{4\pi} ada - \frac{1}{2\pi} ad\hat{A}_J = -\frac{1}{4\pi} \hat{A}_J d\hat{A}_J
\]

\[
\mathcal{L}_{\text{eff}}^{\Pi} = 0
\]

by definition and because duality requires a **single** critical point: this **must** hold throughout the respective phases

equivalently:

\[
m_{f_+} > 0, m_{f_-} < 0
\]

must likewise hold throughout the respective phases
naturalness

we are only able to argue that as $\delta m \to 0$, $f_+ \to \psi, f_- \to \lambda$

• the alternative fermion-mass "evolution" requires fine tuning

• perturbation theory in $\hat{D}_J$ implies

$$m_\psi \propto -\hat{\sigma}_J \Theta(-\hat{\sigma}_J) + \hat{D}_J, m_\lambda < 0$$

consistent with the absence of a fermion gap closing at the transition

\[ \begin{array}{ll}
  m_{u}^2 < 0 & m_{u}^2 > 0 \\
  m_{f_+} > 0 & m_{\psi} > 0 \\
  m_{f_-} < 0 & m_{\lambda} < 0 \\
  m_{\sigma}^2 > 0 & m_{\sigma}^2 > 0
\end{array} \]
interpreting the masses in terms of parameters of the UV theory B: identical phase structure and one zero crossing at the transition

only \( m_u^2 = 0 \) at the transition!
theory B

\[ m^2_u < 0 \]
\[ m_{f+} > 0 \]
\[ m_{f-} < 0 \]
\[ m^2_\sigma > 0 \]

\[ m^2 > 0 \]
\[ m_\psi > 0 \]
\[ m_\lambda < 0 \]
\[ m^2_\sigma > 0 \]
theory B

\[ \hat{\mathcal{D}}_J > 0 \]

\[ \mathcal{L}^{(B)}_{\text{eff}} = |D_{-\alpha} u|^2 - |u|^4 + \frac{1}{4\pi} ada - \frac{1}{2\pi} \hat{\mathcal{A}} da \]

\[ m_u^2 < 0 \quad m_u^2 > 0 \]

\[ m_f^2 > 0 \quad m_f^2 > 0 \]

\[ m_f^- < 0 \quad m_f^- < 0 \]

\[ m_\sigma^2 > 0 \quad \begin{array}{l}
\text{II} \\
\text{I}
\end{array} \quad m_\sigma^2 > 0 \]

\[ m_\psi > 0 \quad m_\psi > 0 \]

\[ m_\lambda < 0 \quad m_\lambda < 0 \]

\[ -\text{sign}(\hat{\sigma}_J) \hat{\sigma}_J^2 \]

III
the critical theories are IR dual

a free fermion is dual to a scalar coupled to a $U(1)_1$ photon

$\bar{\Psi}i\not{D}_A\Psi - \frac{1}{8\pi}\hat{A}d\hat{A} \leftrightarrow |D_{-a}u|^2 - |u|^4 + \frac{1}{4\pi}ada - \frac{1}{2\pi}\hat{A}da$

$m_\Psi > 0$  $m_\Psi < 0$

$m_{\Psi}^2 < 0$  $m_{\Psi}^2 > 0$

$II$  $I$  $II$  $I$
Where’s the fermion in the scalar QED3?

\[ \bar{\Psi} i \not{\partial} \hat{A} \Psi - \frac{1}{8\pi} \hat{A}d\hat{A} \leftrightarrow |D_{-a}u|^2 - |u|^4 + \frac{1}{4\pi} ada - \frac{1}{2\pi} \hat{A}da \]

the fermion should correspond to a monopole-boson composite

\[ \Psi^\dagger \sim M^\dagger u \]

Jackiw & Rebbi; Borokhov, Kapustin, & Wu

(you can check the charge assignment is correct from the CS terms)

this is known as “flux attachment“ in the condensed matter literature

Girvin & MacDonald; Read; Jain(1989); Hansson, Kivelson, & Zhang; Lopez & Fradkin; Halperin, Lee, & Read; Kalmeyer & Zhang

from D. Arovas’ Ph.D thesis
to argue for additional dualities, it’s necessary to stabilize the scalar of theory A in order to access $U(1)_J$ symmetry-broken phases

$$\mathcal{L}^{(A)} = |D_{A_J} v|^2 - m_v^2 |v|^2 + i \bar{\Psi} \slashed{D}_{A_J} \Psi - m_\Psi \bar{\Psi} \Psi - \frac{1}{8\pi} \hat{A}_J d \hat{A}_J$$

$$m_v^2 = \hat{\sigma}_J^2 + \hat{D}_J,$$

$$m_\Psi = \hat{\sigma}_J$$
stabilization

using our “usual” tricks, we introduce $\hat{D}_J$, make it dynamical $\hat{D}_J \rightarrow D_J$ and integrate over $D_J$ with Gaussian weight

$$\mathcal{L}_{D_J} = \frac{1}{2\hbar} (D_J - \hat{D}_0)^2$$

dthis procedure produces the classical scalar potential:

$$\delta \mathcal{L}^{(A)} = (\hat{\sigma}_J + \hat{D}_0) |v|^2 + \frac{\hbar^2}{2} |v|^4$$

the “unstable” regime $m_v^2 = \hat{\sigma}_J + \hat{D}_0 < 0$ now becomes a regime of broken symmetry

$$\langle v \rangle \neq 0$$
theory A phase diagram

\[ \begin{align*}
&m_v^2 > 0 & m_v^2 > 0 \\
&m_\Psi > 0 & m_\Psi < 0 \\
&-\text{sgn}(\hat{\sigma}_J)\hat{\sigma}_J^2
\end{align*} \]
theory B phase diagram

duality implies a phase diagram with identical topology

there is a unique choice of mass assignments!
theory A
phase diagram

\[ \begin{align*}
\text{free fermion} & \\
\hat{D}_0 & \\
m_v^2 > 0 & m_v^2 > 0 \\
m_\Psi > 0 & m_\Psi < 0 \\
\end{align*} \]

\[ -\text{sgn}(\hat{\sigma}_J)\hat{\sigma}_J^2 \]

Wilson-Fisher

\[ \begin{align*}
m_v^2 < 0 & \\
m_\Psi > 0 & \\
\end{align*} \]

theory B
phase diagram

\[ \begin{align*}
\text{sQED} & \\
\hat{D}_0 & \\
m_u^2 < 0 & m_u^2 > 0 \\
m_f^+ > 0 & m_\psi > 0 \\
m_f^- < 0 & m_\lambda < 0 \\
\end{align*} \]

\[ -\text{sgn}(\hat{\sigma}_J)\hat{\sigma}_J^2 \]

sQED

\[ \begin{align*}
m_u^2 > 0 & \\
m_\psi < 0 & \\
m_\lambda > 0 & m_\lambda < 0 \\
\end{align*} \]

fQED
Summary: mirror symmetry implies the dualities:

A free fermion is dual to a scalar coupled to a $U(1)_1$ photon

\[ \bar{\Psi} i \gamma_\hat{A} \hat{A} \Psi - \frac{1}{8\pi} \hat{A} d \hat{A} \leftrightarrow |D_{-a} u|^2 - |u|^4 + \frac{1}{4\pi} a d a - \frac{1}{2\pi} \hat{A} d a \]

A Wilson-Fisher boson is dual to fermion coupled to a $U(1)_{1/2}$ photon

\[ |D_{\hat{A}} v|^2 - |v|^4 - \frac{1}{4\pi} \hat{A} d \hat{A} \leftrightarrow \bar{\Psi} i \gamma_{-a} \psi + \frac{1}{8\pi} a d a - \frac{1}{2\pi} a d \hat{A} \]

A Wilson-Fisher boson is dual to scalar coupled to a $U(1)_0$ photon

\[ |D_{\hat{A}} v|^2 - |v|^4 \leftrightarrow |D_{-a} u|^2 - |u|^4 - \frac{1}{2\pi} a d \hat{A} - \frac{1}{4g^2} f_a^2 \]
Thank you!

(I’ll now describe an application of these dualities)
(free and pure) electrons in a magnetic field

\[ H_0 = \frac{1}{2m_e} (\partial_j - iA_j)^2, \quad \epsilon_n = \frac{B}{m_e} (n + \frac{1}{2}) \]

\[ \partial_x A_y - \partial_y A_x = B > 0 \]

\[ H_0 |m, n\rangle = \epsilon_n |m, n\rangle, \quad m = 1, \ldots, \frac{BA}{2\pi}, \quad n = 0, 1, \ldots \]
particle-hole transformation in the lowest Landau level (LLL)

\[ H = H_0 + H_{\text{int}} \]

low-energy physics is dominated by the LLL when \( \frac{B}{m_e} \to \infty \):

\[ H \to H_{\text{LLL}} = V_{abcd} c_a^\dagger c_b c_c^\dagger c_d - \mu c_a^\dagger c_a + \text{const} \]

\( c_a^\dagger \) creates a state in the LLL orbital labeled by \( a = 1, \ldots, \frac{BA}{2\pi} \)

**PH transformation**: a partially filled lowest Landau level (LLL) can equally well be described by the holes of the filled Landau level:

\[ c_a^\dagger \mapsto c_a^{\text{hole}} \]

it’s our choice for what’s the most convenient set of “coordinates”

\( \nu_e = 0 : \) insulator of electrons \( \leftrightarrow \) IQHE of holes

\( \nu_e = 1 : \) IQHE of electrons \( \leftrightarrow \) insulator of holes
electrons at half-filling: Dirac composite fermion

singular LLL $\frac{B}{m_e} \rightarrow \infty$ is “regularized” by a Dirac electron lagrangian

“fermionic particle-vortex” duality then gives the Dirac CF dual

$$\bar{\Psi} e^{i(\partial_\mu - iA_\mu)\gamma^\mu \Psi} + \frac{1}{8\pi} e^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \leftrightarrow \bar{\psi} e^{i(\partial_\mu - i\alpha_\mu)\gamma^\mu \psi} - \frac{1}{4\pi} e^{\mu\nu\rho} \alpha_\mu \partial_\nu A_\rho + \frac{1}{8\pi} e^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$$

(the Dirac CF lagrangian is written in a simplified gauge-fixed form)

attractive features:

(i) the PH transformation is realized as time-reversal + a modular T-transformation (more on this momentarily) of the Dirac CF formulation and is manifestly unbroken

(ii) 1-parameter scaling of the conductivity tensor

question:

How can the “fermionic particle-vortex” duality be derived?
SL(2,Z) action on 2+1D CFTs with U(1) global symmetry

\[ \mathcal{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \]

\[ \mathcal{S} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \]

for example,

\[ \mathcal{T} : z \mapsto \mathcal{T} \begin{pmatrix} z \\ 1 \end{pmatrix} \]
SL(2,Z) action on 2+1D CFTs with U(1) global symmetry
SL(2,Z) action on 2+1D CFTs with U(1) global symmetry

\[ \mathcal{T} : \mathcal{L}(\Phi, \hat{A}) \mapsto \mathcal{L}(\Phi, \hat{A}) + \frac{1}{4\pi} \hat{A} d \hat{A}, \]

\[ \mathcal{S} : \mathcal{L}(\Phi, \hat{A}) \mapsto \mathcal{L}(\Phi, a) - \frac{1}{2\pi} \hat{B} d a. \]

this action induces a modular transformation on the complexified conductivity \( \sigma = \sigma_{xy} + i\sigma_{xx} \)
application of a particular SL(2,\mathbb{Z}) transformation on the bosonization
dualities gives the fermion/fermion duality

$$|D_A v|^2 - |v|^4 - \frac{1}{4\pi} \hat{A}d\hat{A} \leftrightarrow \bar{\psi} i \not{\partial}_{-a} \psi + \frac{1}{8\pi} ada - \frac{1}{2\pi} ad\hat{A}$$

$$\downarrow \quad \mathcal{S}\mathcal{T}^2$$

$$|D_{-a} v|^2 - |v|^4 + \frac{1}{4\pi} ada - \frac{1}{2\pi} ad\hat{A} \leftrightarrow \bar{\psi} \not{\partial}_{-b} \psi + \frac{1}{4\pi} bdb + \frac{1}{2\pi} cdb + \frac{2}{4\pi} cdc + \frac{1}{2\pi} \hat{A}dc$$

using a bosonization duality

$$\downarrow$$

$$\bar{\Psi} \not{\partial}_{\hat{A}} \Psi - \frac{1}{8\pi} \hat{A}d\hat{A} \leftrightarrow \bar{\psi} \not{\partial}_{-b} \psi + \frac{1}{4\pi} bdb + \frac{1}{2\pi} cdb + \frac{2}{4\pi} cdc + \frac{1}{2\pi} \hat{A}dc$$

thus, the dualities derived from mirror symmetry underlie the
fermion/fermion duality written in the last line above
Thank you!

(for real this time)